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SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

By E. M. Coon and R. L. Gordon.

In the process of finding the singular solutions of differential equations of the first order and also in finding the envelopes of families of curves with one parameter, some extraneous loci appear. Analogous ones appear in the theory of singular solutions of differential equations of higher order, and of osculating curves of families of curves with more than one parameter. In discussions concerning singular solutions of differential equations of higher order attention is usually called to a locus of cusps of the second kind, and to a singular solution of the second kind, which has no analogue for differential equations of the first order, but we have found no mention or illustrations of a locus of cusps of the first kind, or of a tac-locus, where non-consecutive curves have contact of higher order. In this paper we give a few examples to illustrate cases that may arise for differential equations of the second order. We also consider the theory of osculating curves of families of curves with two parameters, and illustrate the fact that besides an osculating curve, a cusp locus or a nodal locus may appear.

In the first example the differential equation has two singular solutions, which give the osculating curves with contact of the second order with the family of integrals. In the second example there is a singular solution, and a locus of cusps of the first kind with an infinite number of cusps at each point. In the third example there is a tac-locus with contact of the second order, a locus of cusps of the first kind with an infinite number of cusps at each point, and in connection with the osculating curves a nodal-locus with an infinite number of nodes at each point. In the fourth example there is a singular solution and a locus of cusps of the first kind with one cusp at each point.

Let

$$F(x, y, y', y'') = 0 \tag{1}$$

be a differential equation of the second order, where F is a polynomial in x, y, y', y'' which cannot be broken up into factors, and is of degree $n (n > 1)$ in y''. A singular solution of this equation is a solution which is not contained in the general solution. Such a solution, if it exists, must* satisfy $\varphi_1(x, y, y') = 0$, a differential equation of the first order obtained by forming the *resultant* of (1) and

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* Forsyth, Theory of differential equations, part II, vol. III, chapter 15.

$$\frac{\partial F}{\partial y''} = 0 \tag{2}$$

to eliminate y''. The general solution of $\varphi_1 = 0$ is a *singular solution of the first kind* of $F = 0$, if it is a solution of $F = 0$, and a singular solution of $\varphi_1 = 0$ which satisfies $F = 0$, is a *singular solution of the second kind* of $F = 0$. Let

$$\varphi_2(x, y, y') = 0$$

be the differential equation of first order obtained by the elimination of y'' between (1) and

$$\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} = 0 \tag{3}$$

A necessary condition* for the existence of a singular solution of $F = 0$ is that (1), (2) and (3) have a common solution in y'', or a necessary condition that $y = s(x)$ be a singular solution of $F = 0$, is that $y = s(x)$ satisfy $\varphi_1 = 0$ and $\varphi_2 = 0$. A sufficient condition for the existence of a singular solution is that (1), (2) and (3) have a common solution in y'', when $(\partial F/\partial x) + y'(\partial F/\partial y)$ and $(\partial F/\partial y')$ are not also zero for that value of y''.

In addition to the singular solution, $\varphi_1(x, y, y') = 0$ may yield a locus of cusps of either the first or second kinds, a tac-locus on which two non-consecutive curves have contact of second order, or a particular solution. In addition to the singular solution, $\varphi_2(x, y, y'') = 0$ may contain a locus of points where $y''' = 0$, a tac-locus on which two non-consecutive curves have contact of second order, or a particular solution.

The integral curves of $\varphi_1(x, y, y') = 0$, which are singular solutions, and the integral curves $f(x, y, a, b) = 0$ of $F(x, y, y', y'') = 0$ have contact of second order, that is, the integral curves of $\varphi_1 = 0$ are osculating curves of the integral curves of $F = 0$.

Let $f(x, y, a, b) = 0$ be the equation of a two-parameter family of curves. There may exist a one-parameter family of curves which has contact of second order with the given family of curves. It can be shown, in a manner similar to the method of finding the envelope of a one parameter family of curves, that the osculating curves satisfy the differential equation of first order $\varphi(x, y, y') = 0$, found by eliminating a,b, and db/da from the following four equations

$$f(x, y, a, b) = 0,$$

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* Goursat, E., Sur les solutions singulières des équations différentielles simultanées, pp. 362–366, American Journal of Mathematics, vol. XI, 1888–89.

$$\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} = 0,$$

$$\frac{\partial f}{\partial a} + b'\frac{\partial f}{\partial b} = 0,$$

$$\frac{\partial^2 f}{\partial x \partial a} + b'\frac{\partial^2 f}{\partial x \partial b} + y'\left(\frac{\partial^2 f}{\partial y \partial a} + b'\frac{\partial^2 f}{\partial y \partial b}\right) = 0.$$

Besides the osculating curves, $\varphi(x, y, y') = 0$ may contain a locus of cusps of either the first or second kinds, or a nodal locus.

Example 1.

$$F = y''^3 - 4xy'y'' + 8y'^2 = 0,$$

$$f = (y - b) - a\frac{(x - a)^3}{3} = 0,$$

$$\varphi_1 = y'^3\left(y' - \frac{4x^3}{27}\right) = 0,$$

$$\varphi_2 = y'^4\left(y' - \frac{4x^3}{27}\right) = 0,$$

$$\varphi = y'\left(y' - \frac{4x^3}{27}\right) = 0,$$

$y' = 0$ and $y' - 4x^3/27 = 0$ satisfy $\varphi_1 = 0$, $\varphi_2 = 0$, and $\varphi = 0$; their integrals $y = c_1$, and $y - x^4/27 = c_2$ are singular solutions of the first kind of $F = 0$, and $y = c_1$ is also a particular solution.

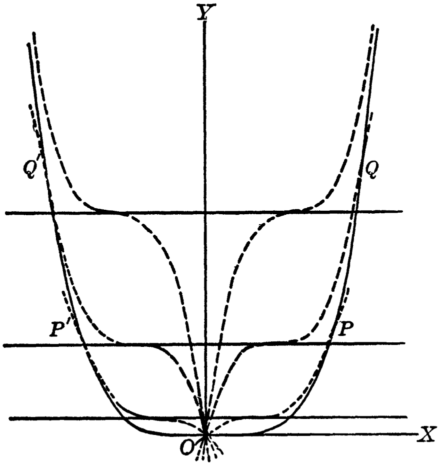

FIG. 1.

The dotted curves in Fig. 1 represent the integral curves, $f = 0$, obtained by giving the parameter a different values and the parameter

b the value zero. The equation of the curve $Q'P'OPQ$ is $y = x^4/27$; it is an osculating curve of the integral curves ($b = 0$), with contact of the second order. The straight lines $y = c_1$ are also osculating curves of the integral curves with contact of the second order at the points of inflection. Translation parallel to the Y-axis gives the complete system of the integral curves and their osculating curves.

Example 2.

$$F = 4xy''^2 + 2xy'' - y' = 0,$$

$$f = (y - b - ax) \pm \tfrac{2}{3}a^{1/2}x^{3/2} = 0,$$

$$\varphi_1 = x^2\left(y' + \frac{x}{4}\right) = 0,$$

$$\varphi_2 = y'\left(y' + \frac{x}{4}\right) = 0,$$

$$\varphi = x^3\left(y' + \frac{x}{4}\right) = 0.$$

$x = 0$ satisfies $\varphi_1 = 0$ and $\varphi = 0$, but does not satisfy $F = 0$; it is a cusp-locus. $y' + x/4 = 0$ satisfies $\varphi_1 = 0$, $\varphi_2 = 0$ and $\varphi = 0$, and the integrals $y + x^2/8 = c_1$ are singular solutions of $F = 0$. $y' = 0$ satisfies only $\varphi_2 = 0$, and the integrals $y_1 = c_2$ are particular solutions of $F = 0$.

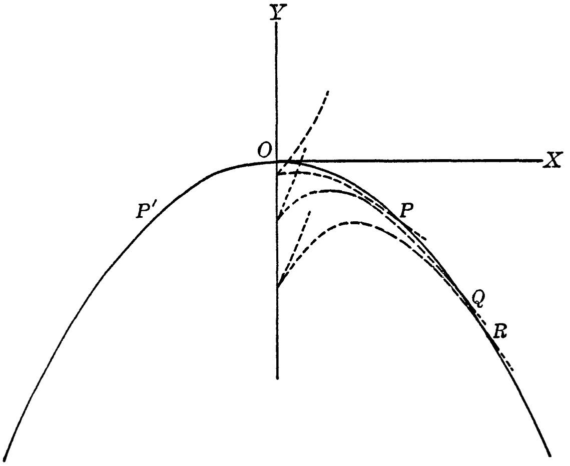

FIG. 2.

The dotted curves in Fig. 2 represent the integral curves, $f = 0$, obtained by giving the parameter a different values, and the parameter b the value zero. The equation of the curve $P'OPQR$ is $y + x^2/8 = 0$;

it is an osculating curve of the family ($b = 0$) with contact of the second order. The Y-axis is a locus of cusps of the first kind. Translation parallel to the Y-axis gives the complete system; there is an infinite number of cusps at each point of the Y-axis.

Example 3.

$$F = yy''^2 - y'^6(y - 1)^2 = 0,$$

$$f = 3(x + ay - b) \pm \tfrac{4}{5}\, y^{3/2}(y - 5) = 0,$$

$$\varphi_1 = y^2y'^6(y - 1)^2 = 0,$$

$$\varphi_2 = y'^{14}(y - 1)^2[(y + 1)^2 - 36y\, y'^2(y - 1)^4] = 0,$$

$$\varphi = y^3(y - 5)^2 = 0.$$

$y - 5 = 0$ satisfies only $\varphi = 0$; it is a nodal locus. $y - 1 = 0$ satisfies $\varphi_1 = 0$ and $\varphi_2 = 0$; it is a tac-locus. $y = 0$ satisfies $\varphi_1 = 0$ and $\varphi = 0$; it is a cusp-locus. $[(y + 1)^2 - 36y\, y'^2(y - 1)^4] = 0$ satisfies $\varphi_2 = 0$; it gives a locus of points where $y''' = 0$. $y' = 0$ satisfies $\varphi_1 = 0$ and $\varphi_2 = 0$; the integrals $y = c$ are particular solutions of $F = 0$.

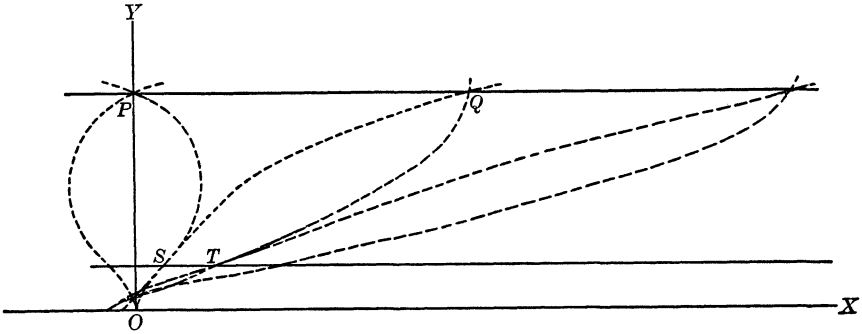

FIG. 3.

The dotted curves in Fig. 3 represent the integral curves, $f = 0$, obtained by giving the parameter a different values, and the parameter b the value zero. Translation parallel to the X-axis gives the complete system of integral curves. The equation of the straight line PQ is $y = 5$; there is an infinite number of nodes at each point. The equation of the straight line ST is $y = 1$; at each point there is an infinite number of pairs of non-consecutive curves with contact of the second order. At each point of the X-axis there is an infinite number of cusps of the first kind.

Example 4.

$$F = 8(y'' - 2)^2 y' - 9(y'' - 2) - 9 = 0,$$

$$f = y - b - (x - a)^2 \pm (x - a)^{3/2} = 0,$$

$$\varphi_1 = y'(32y' + 9) = 0,$$

$$\varphi_2 = (32y' + 9) = 0,$$

$$\varphi = y'^8(32y' + 9) = 0.$$

$y' = 0$ satisfies $\varphi_1 = 0$ and $\varphi_2 = 0$; the integrals $y = c_1$ are cusp-loci. $32y' + 9 = 0$ satisfies $\varphi_1 = 0$, $\varphi_2 = 0$, and $\varphi = 0$, and the integrals $y + 9x/32 = c_2$ are singular solutions of the first kind of $F = 0$.

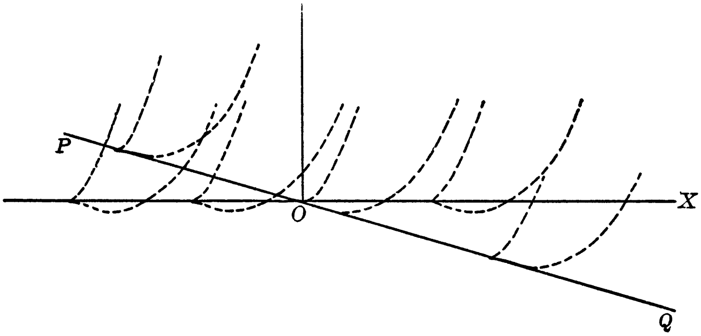

FIG. 4.

The dotted curves in Fig. IV represent the integral curves $f = 0$. The X-axis and lines parallel to it are loci of cusps of the first kind. The equation of the line PQ is $y + 9x/32 = 0$. The line PQ and lines parallel to it are osculating curves of the integral curves $f = 0$.

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